

# Mutually algebraic structures and expansions by predicates

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## Abstract

We introduce the notions of a mutually algebraic structures and theories and prove many equivalents. A theory  $T$  is mutually algebraic if and only if it is weakly minimal and trivial if and only if no model  $M$  of  $T$  has an expansion  $(M, A)$  by a unary predicate with the finite cover property. We show that every structure has a maximal mutually algebraic reduct, and give a strong structure theorem for the class of elementary extensions of a fixed mutually algebraic structure.

## 1 Introduction

This paper is written with two objectives in mind. On one hand, it is a continuation of [5], where a strong quantifier elimination theorem was proved for elementary diagrams of models of a weakly minimal, trivial theory. Here, we show that the crucial notion of mutual algebraicity of a formula (see Definition 2.2) has meaning in arbitrary structures, and in fact describes a specific reduct of any structure. As well, Theorem 3.3 reverses the argument in [5]. The quantifier elimination result described there can only occur as the elementary diagram of a weakly minimal, trivial theory.

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On the other hand, there has been a large body of research about whether an expansion  $(M, A)$  of a given stable structure  $M$  by a unary predicate  $A$  remains stable. Sufficient conditions abound, but the general question remains open. Here, also with Theorem 3.3, we characterize those structures  $M$  with the property that every unary expansion  $(M, A)$  satisfies the non-finite cover property (nfc), which is a strengthening of stability.

The motivation for this came from the author's reading [1], where Baldwin and Baizhanov showed that a non-trivial, strongly minimal structure  $M$  has an unstable expansion  $(M, A)$ . Thanks are due to John Baldwin for a careful reading of this paper, and for pointing out that an alternate treatment of a portion of Section 4 appears in Section 6 of [2].

## 2 The mutually algebraic reduct of a structure

We begin by recalling the definition of a mutually algebraic formula. This notion was introduced by Dolich, Raichev, and the author in [4] and further developed in [5]. However, in both of those papers, the ambient theory was assumed to have the non-finite cover property (nfc). Here, we define the notions without any ambient assumptions. We begin by formulating the notion of a mutually algebraic set.

**Definition 2.1** Given an arbitrary set  $A$  and an integer  $n \geq 1$ , a *proper partition of  $n$*  is a partition  $X \sqcup Y = \{1, \dots, n\}$  where  $X, Y$  are disjoint and each is non-empty. Given such a partition,  $\pi_Y$  denotes the projection of  $A^n$  onto the coordinates in  $Y$ .

A subset  $B \subseteq A^n$  is *mutually algebraic* if there is a number  $K$  so that for any proper partition of the coordinates  $X \sqcup Y = \{1, \dots, n\}$ , the projection  $\pi_Y$  restricted to  $B$  is at most  $K$ -to-1. That is,  $|\pi_Y^{-1}(\bar{b}_Y) \cap B| \leq K$  for any  $\bar{b}_Y \in \pi_Y(B)$ .

As special cases, note that if either  $A$  is finite or  $B$  is empty, then  $B$  is mutually algebraic. Furthermore, for any set  $A$ , every subset  $B \subseteq A^1$  is mutually algebraic as there are no proper partitions of a one element set.

**Definition 2.2** Let  $M$  denote any  $L$ -structure. An  $L(M)$ -formula  $\varphi(\bar{z})$  is *mutually algebraic* if  $\varphi(M) := \{\bar{a} \in M^{\text{lg}(\bar{z})} : M \models \varphi(\bar{a})\}$  is a mutually

algebraic subset of  $M^{\text{lg}(\bar{z})}$ . We let  $\mathcal{MA}(M)$  denote the set of all mutually algebraic  $L(M)$ -formulas. When  $M$  is understood, we simply write  $\mathcal{MA}$ .

To clarify this concept and to set notation, given a formula  $\varphi(\bar{z})$ , a *proper partition* of  $\bar{z}$  has the form  $\bar{z} = \bar{x} \hat{\ } \bar{y}$ , where  $\bar{x}, \bar{y}$  are disjoint and  $\text{lg}(\bar{x}), \text{lg}(\bar{y}) \geq 1$ . We do not require  $\bar{x}$  be an initial segment of  $\bar{z}$  but to simplify notation, we write it as if it were. Then, for any  $L$ -structure  $M$ , an  $L(M)$ -formula  $\varphi(\bar{z})$  is mutually algebraic if and only if there is an integer  $K$  so that  $M \models \forall \bar{y} \exists^{\leq K} \bar{x} \varphi(\bar{x}, \bar{y})$  for every proper partition  $\bar{x} \hat{\ } \bar{y}$  of  $\bar{z}$ .

The reader is cautioned that whether a formula  $\varphi(\bar{z})$  is mutually algebraic or not depends on the choice of free variables. In particular, mutual algebraicity is **not** preserved under adjunction of dummy variables. The special cases mentioned above imply that if  $M$  is finite, then every  $L(M)$ -formula is in  $\mathcal{MA}(M)$ , and for an arbitrary  $M$ , every inconsistent formula and every  $L(M)$ -formula  $\varphi(z)$  with exactly one free variable symbol is mutually algebraic. Our first easy Lemma gives a semantic interpretation to this notion when  $\text{lg}(\bar{z}) \geq 2$ :

**Lemma 2.3** *Let  $M$  be any  $L$ -structure. The following are equivalent for any  $L(M)$ -formula  $\varphi(\bar{z})$  with  $\text{lg}(\bar{z}) \geq 2$ :*

1.  $\varphi(\bar{z}) \in \mathcal{MA}(M)$ ;
2. *There is an integer  $K$  so that  $M \models \forall x \exists^{\leq K} \bar{y} \varphi(x, \bar{y})$  for all partitions  $\bar{z} = x \hat{\ } \bar{y}$  with  $\text{lg}(x) = 1$ ;*
3. *For all  $N \succeq M$ , for all  $\bar{e} \in N^{\text{lg}(\bar{z})}$  realizing  $\varphi$ , and for all  $e \in \bar{e}$ ,  $\bar{e} \subseteq \text{acl}(M \cup \{e\})$  (i.e., every  $e' \in \bar{e}$  is in  $\text{acl}(M \cup \{e\})$ ).*

**Proof.** (1)  $\Rightarrow$  (2) is immediate.

(2)  $\Rightarrow$  (3) Fix any  $N \succeq M$  and assume  $N \models \varphi(\bar{e})$ . Fix any variable symbol  $x \in \bar{z}$  and let  $e$  be the corresponding element of  $\bar{e}$ . By elementarity,  $N \models \exists^{\leq K} \bar{y} \varphi(e, \bar{y})$ , so  $\bar{e} \subseteq \text{acl}(M \cup \{e\})$ .

(3)  $\Rightarrow$  (1) If (1) fails, then for some proper partition  $\bar{z} = \bar{x} \hat{\ } \bar{y}$  we have  $M \models \exists \bar{y} \exists^{\geq r} \bar{x} \varphi(\bar{x}, \bar{y})$ . Thus, by compactness, there is  $N \succeq M$  and  $\bar{b}$  from  $N$  such that  $N \models \exists^{\geq r} \bar{x} \varphi(\bar{x}, \bar{b})$  for each  $r \in \omega$ . By compactness again, there is  $N^* \succeq N$  and  $\bar{a} \in (N^*)^{\text{lg}(\bar{x})}$  such that  $\bar{a} \not\subseteq \text{acl}(M \cup \bar{b})$ , contradicting (3).

The following Lemma indicates some of the closure properties of the set  $\mathcal{MA}$ . In what follows, when we write  $\varphi(\bar{x}, \bar{y}) \in \mathcal{MA}$ , we mean that  $\bar{x}$  and  $\bar{y}$  are disjoint sets of variable symbols and  $\varphi(\bar{z}) \in \mathcal{MA}$  where  $\bar{z} = \bar{\hat{x}}\bar{y}$ , but that we are concentrating on a specific proper partition of  $\varphi(\bar{z})$ .

**Lemma 2.4** *Let  $M$  be any structure in any language  $L$ .*

1. *If  $\varphi(\bar{z}) \in \mathcal{MA}$ , then  $\varphi(\sigma(\bar{z})) \in \mathcal{MA}$  for any permutation  $\sigma$  of the variable symbols;*
2. *If  $\varphi(\bar{x}, \bar{y}) \in \mathcal{MA}$  and  $\bar{a} \in M^{\text{lg}(\bar{y})}$ , then both  $\exists \bar{y} \varphi(\bar{x}, \bar{y})$  and  $\varphi(\bar{x}, \bar{a}) \in \mathcal{MA}$ ;*
3. *If  $\varphi(\bar{z}) \vdash \psi(\bar{z})$  and  $\psi(\bar{z}) \in \mathcal{MA}$ , then  $\varphi(\bar{z}) \in \mathcal{MA}$ ;*
4. *If  $\{\varphi_i(\bar{z}_i) : i < k\} \subseteq \mathcal{MA}$ , and there is some variable  $x$  common to every  $\bar{z}_i$ , then  $\psi(\bar{w}) := \bigwedge_{i < k} \varphi_i(\bar{z}_i) \in \mathcal{MA}$ , where  $\bar{w} = \bigcup_{i < k} \bar{z}_i$ ;*
5. *If  $\varphi(\bar{x}, \bar{y}) \in \mathcal{MA}$  and  $r \in \omega$ , then  $\theta_r(\bar{y}) := \exists^{\geq r} \bar{x} \varphi(\bar{x}, \bar{y}) \in \mathcal{MA}$ .*

**Proof.** The verification of (1), (2), and (3) are immediate. Concerning (4), we apply Lemma 2.3. Fix  $N \succeq M$  and  $\bar{e}$  such that  $N \models \psi(\bar{e})$ . Let  $x$  denote a variable symbol that appears in every  $\bar{z}_i$  and let  $e_x$  denote the element of  $\bar{e}$  corresponding to  $x$ . Similarly, for each  $i < k$  let  $\bar{e}_i$  be the subsequence corresponding to  $\bar{z}_i$ . As each  $\varphi_i(\bar{z}_i) \in \mathcal{MA}$ ,  $e_x \in \text{acl}(M \cup \{e\})$  for every  $e \in \bar{e}_i$ , so  $e_x \in \text{acl}(M \cup \{e\})$  for every  $e \in \bar{e}$ . But also,  $e \in \text{acl}(M \cup \{e_x\})$  for every  $e \in \bar{e}$ . Thus, by the transitivity of algebraic closure,  $e \in \text{acl}(M \cup \{e'\})$  for all pairs  $e, e' \in \bar{e}$ . So  $\psi(\bar{w}) \in \mathcal{MA}$  by Lemma 2.3.

To establish (5), let  $\{\bar{x}_i : i < r\}$  be disjoint sequences of variable symbols, each disjoint from  $\bar{y}$ . Then  $\theta_r(\bar{y})$  is equivalent to

$$\exists \bar{x}_0 \exists \bar{x}_1 \dots \exists \bar{x}_{r-1} \left( \bigwedge_{i < r} \varphi(\bar{x}_i, \bar{y}) \wedge \bigwedge_{i < j < r} \bar{x}_i \neq \bar{x}_j \right)$$

That this formula is in  $\mathcal{MA}$  follows by successively applying Clauses (4), (3), and (2).

**Definition 2.5** For any  $L$ -structure  $M$ , let  $M_M$  denote the canonical expansion of  $M$  to an  $L(M)$ -structure formed by adding a constant symbol  $c_a$  for each  $a \in M$ . We let  $\mathcal{MA}^*(M)$  denote the set of all  $L(M)$ -formulas that are  $Th(M_M)$ -equivalent to a boolean combination of formulas from  $\mathcal{MA}(M)$ . When  $M$  is understood, we simply write  $\mathcal{MA}^*$ .

Whereas the definition of  $\mathcal{MA}$  was rather fussy, membership in  $\mathcal{MA}^*$  is more relaxed, mostly owing to the fact that  $\mathcal{MA}^*$  is closed under adjunction of dummy variables. Indeed, we will see with Proposition 2.7 below, for any structure  $M$ ,  $\mathcal{MA}^*(M)$  specifies a reduct of the canonical expansion  $M_M$ .

**Lemma 2.6** *Let  $M$  denote any  $L$ -structure.*

1.  $\mathcal{MA}^*$  is closed under boolean combinations;
2.  $\mathcal{MA}^*$  is closed under adjunction of dummy variables, i.e., if  $\varphi(\bar{z}) \in \mathcal{MA}^*$  then  $\varphi(x, \bar{z}) \in \mathcal{MA}^*$ ;
3. For each  $k \geq 1$ , if  $\{\varphi_i(x, \bar{y}_i) : i < k\} \subseteq \mathcal{MA}$  and  $r \in \omega$ , then each of  $\exists^{=r} x \bigvee_{i < k} \varphi_i(x, \bar{y}_i)$ ,  $\exists^{\leq r} x \bigvee_{i < k} \varphi_i(x, \bar{y}_i)$ , and  $\exists^{\geq r} x \bigvee_{i < k} \varphi_i(x, \bar{y}_i)$  are in  $\mathcal{MA}^*$ .

**Proof.** The proof of (1) is immediate. For (2), note that  $\psi(x) := 'x = x'$  is in  $\mathcal{MA}$ , hence in  $\mathcal{MA}^*$ , but  $\varphi(x, \bar{z})$  is equivalent to  $\varphi(\bar{z}) \wedge \psi(x)$ . The verification of (3) is more substantial. We argue by induction on  $k$  that for every  $r \in \omega$ ,  $\exists^{=r} x \bigvee_{i < k} \varphi_i(x, \bar{y}_i) \in \mathcal{MA}^*$  for every  $k$ -element subset  $\{\varphi_i(x, \bar{y}_i) : i < k\}$  from  $\mathcal{MA}$ . This suffices, as  $\mathcal{MA}^*$  is closed under boolean combinations and the trivial facts that  $\exists^{\leq r} x \theta$  is equivalent to  $\bigvee_{s \leq r} \exists^{=s} x \theta$  and  $\exists^{\geq r} x \theta$  is equivalent to  $\neg \exists^{\leq r-1} x \theta$ .

To handle the case when  $k = 1$ , fix any  $\varphi(x, \bar{y}) \in \mathcal{MA}$  and any  $r \in \omega$ . By Lemma 2.4(5), both  $\exists^{\geq r} x \varphi(x, \bar{y}) \in \mathcal{MA}$  and  $\exists^{\geq r+1} x \varphi(x, \bar{y}) \in \mathcal{MA}$  and  $\exists^{=r} x \varphi(x, \bar{y})$  is a boolean combination of these.

Next, inductively assume that for every  $r \in \omega$ ,  $\exists^{=r} x \bigvee_{i < k} \varphi_i(x, \bar{y}_i) \in \mathcal{MA}^*$  for every  $k$ -element subset  $\{\varphi_i(x, \bar{y}_i) : i < k\}$  from  $\mathcal{MA}$ . Choose any  $(k+1)$ -element subset  $\{\varphi_i(x, \bar{y}_i) : i \leq k\}$  from  $\mathcal{MA}$  and choose any  $r \in \omega$ . As notation, let  $\psi(x, \bar{w}) := \bigvee_{i < k} \varphi_i(x, \bar{y}_i)$ . By the inclusion/exclusion

principle of integers, the formula  $\exists^{=r}x \bigvee_{i \leq k} \varphi(x, \bar{y}_i)$ , which is equivalent to  $\exists^{=r}x(\psi(x, \bar{w}) \vee \varphi_k(x, \bar{y}_k))$ , is equivalent to

$$\bigvee_{\substack{a, b \leq r \\ a+b-c=r}} \left( \exists^{=a}x \psi(x, \bar{w}) \wedge \exists^{=b}x \varphi_k(x, \bar{y}_k) \wedge \exists^{=c}x [\psi(x, \bar{w}) \wedge \varphi_k(x, \bar{y}_k)] \right)$$

By the inductive hypothesis  $\exists^{=a}x \psi(x, \bar{w}) \in \mathcal{MA}^*$  and  $\exists^{=b}x \varphi_k(x, \bar{y}_k) \in \mathcal{MA}^*$  by the case  $k = 1$ . Also, note that  $\psi(x, \bar{w}) \wedge \varphi_k(x, \bar{y}_k)$  is equivalent to  $\bigvee_{i < k} \delta_i(x, \bar{y}_i, \bar{y}_k)$ , where each  $\delta_i(x, \bar{y}_i, \bar{y}_k) := \varphi_i(x, \bar{y}_i) \wedge \varphi_k(x, \bar{y}_k)$  is in  $\mathcal{MA}$  by Lemma 2.4(4). Thus, by applying the inductive hypothesis to this  $k$ -element subset from  $\mathcal{MA}$ , we conclude that  $\exists^{=c}x(\psi(x, \bar{w}) \wedge \varphi_k(x, \bar{y}_k)) \in \mathcal{MA}^*$ , completing the proof.

**Proposition 2.7** *For any structure  $M$ , the set  $\mathcal{MA}^*(M)$  is closed under existential quantification. Thus, the structure with universe  $M$ , together with the definable sets  $\mathcal{MA}^*(M)$ , is a reduct of the canonical expansion  $M_M$ .*

**Proof.** The second sentence follows from the first, since  $\mathcal{MA}^*$  is a set of  $L(M)$ -formulas closed under boolean combinations. To establish the first sentence, there are two cases. First, if the structure  $M$  is finite, then every  $L(M)$ -formula  $\varphi(\bar{z}) \in \mathcal{MA}$ , so  $\mathcal{MA}^*$  is precisely the elementary diagram of  $M$  and there is nothing to prove. So assume that  $M$  is infinite.

Choose  $\varphi(x, \bar{y}) \in \mathcal{MA}^*$  and we argue that  $\exists x \varphi(x, \bar{y})$  is equivalent to a formula in  $\mathcal{MA}^*$ . By writing  $\varphi$  in Disjunctive Normal Form and noting that disjunction commutes with existential quantification, we may assume that  $\varphi(x, \bar{y})$  has the form

$$\bigwedge_{i < k} \beta_i(x, \bar{y}_i) \wedge \bigwedge_{j < m} \neg \gamma_j(x, \bar{y}_j)$$

where each  $\beta_i$  and  $\gamma_j$  are in  $\mathcal{MA}$  and the variable  $x$  occurs in each of these subformulas. By Lemma 2.4(4), if  $k \geq 1$ , then  $\bigwedge_{i < k} \beta_i(x, \bar{y}_i) \in \mathcal{MA}$ , so we may assume there is at most one  $\beta$ . If there is no  $\beta$ , then since the model  $M$  is infinite, then for any choice of  $\bar{y}$ ,  $\exists x \varphi(x, \bar{y})$  always holds. Thus, we assume that there is exactly one  $\beta$ , i.e., that  $\varphi(x, \bar{y})$  has the form  $\beta(x, \bar{y}^*) \wedge \bigwedge_{j < m} \neg \gamma_j(x, \bar{y}_j)$ , where  $\bar{y}^*$  and each  $\bar{y}_j$  are subsequences of  $\bar{y}$ , and both  $\beta$  and each  $\gamma_j$  are from  $\mathcal{MA}$ .

We first consider the case where  $\bar{y}^*$  is empty. In this case, we may additionally assume that no  $\bar{y}_j$  is empty, since we could replace  $\beta(x)$  by  $\beta(x) \wedge \neg\gamma_j(x)$ . Thus, for any choice of  $\bar{y}$ , the solution set of  $\bigwedge_{j < m} \neg\gamma_j(x, \bar{y}_j)$  is a cofinite subset of  $M$ . We have two subcases: On one hand, if  $\beta(x)$  were algebraic, then every solution to  $\beta$  lies in  $M$ , hence  $\varphi(x, \bar{y})$  would be equivalent to  $\bigvee_{m \in \beta(M)} \varphi(m, \bar{y})$ , which would be in  $\mathcal{MA}^*$  by Lemma 2.4(2). On the other hand, if  $\beta(x)$  were non-algebraic, then  $\beta(x)$  would have infinitely many solutions in  $M$ , so  $\varphi(x, \bar{y})$  would have a solution in  $M$  for any choice of  $\bar{y}$ . Thus,  $\exists x \varphi(x, \bar{y})$  would always hold.

Finally, assume that  $\bar{y}^* \neq \emptyset$ . By the definition of mutual algebraicity, there is an integer  $K$  so that  $M \models \forall \bar{y}^* \exists^{\leq K} x \beta(x, \bar{y}^*)$ . For each  $j < m$ , let  $\theta_j(x, \bar{y}^*, \bar{y}_j) := \beta(x, \bar{y}^*) \wedge \gamma_j(x, \bar{y}_j)$ . By Lemma 2.4(4), each  $\theta_j(x, \bar{y}^*, \bar{y}_j) \in \mathcal{MA}$ . Thus, the formula  $\exists x \varphi(x, \bar{y})$  is equivalent to

$$\bigvee_{r \leq K} \left( \exists^= x \beta(x, \bar{y}^*) \wedge \exists^{< r} x \bigvee_{j < m} \theta_j(x, \bar{y}^*, \bar{y}_j) \right)$$

which is in  $\mathcal{MA}^*$  by Lemma 2.6.

The previous Proposition inspires the following two definitions:

**Definition 2.8** A structure  $M$  is *mutually algebraic* if every  $L(M)$ -formula is in  $\mathcal{MA}^*(M)$ .

**Definition 2.9** Let  $M$  be any structure. The *mutually algebraic reduct* of  $M_M$  is the structure with the same universe as  $M$ , and whose definable sets are precisely  $\mathcal{MA}^*(M)$ .

Proposition 2.7 immediately implies that the mutually algebraic reduct of a structure  $M$  is a mutually algebraic structure.

**Lemma 2.10** *Mutual algebraicity of structures is preserved under elementary equivalence.*

**Proof.** Suppose that  $M$  is a mutually algebraic structure and that  $N$  is elementarily equivalent to  $M$ . It suffices to show that  $\varphi(\bar{x}, \bar{h}) \in \mathcal{MA}^*(N)$  for any  $L$ -formula  $\varphi(\bar{x}, \bar{y})$  (with  $\bar{x}$  and  $\bar{y}$  disjoint and there are no hidden

parameters) and any  $\bar{h} \in N^{\text{lg}(\bar{y})}$ . Given this data, let  $\bar{z} = \bar{x} \hat{\ } \bar{y}$  and consider the  $L$ -formula  $\varphi(\bar{z})$ . As  $M$  is mutually algebraic,  $\varphi(\bar{z}) \in \mathcal{MA}^*(M)$ , so there are (finitely many)  $L$ -formulas  $\delta_i(\bar{z}, \bar{w}_i)$  and  $\bar{e}_i$  from  $M$  so that (1)  $\varphi(\bar{z})$  is  $Th(M_M)$ -equivalent to a boolean combination  $\theta(\bar{z}, \bar{e}^*)$  of the  $\delta_i(\bar{z}, \bar{e}_i)$  ( $\bar{e}^*$  denotes the concatenation of the  $\bar{e}_i$ 's); and (2) There is a number  $K$  so that each of the formulas  $\delta_i(\bar{z}, \bar{e}_i)$  satisfy  $M \models \forall \bar{y}' \exists^{\leq K} \bar{x}' \delta_i(\bar{x}', \bar{y}', \bar{e}_i)$  for every proper partition  $\bar{z} = \bar{x}' \hat{\ } \bar{y}'$ . Thus, by quantifying out the  $\bar{e}^*$ , there is an  $L$ -sentence  $\sigma$  asserting that

$$\exists \bar{w}^* \left( \forall \bar{z} [\varphi(\bar{z}) \leftrightarrow \theta(\bar{z}, \bar{w}^*)] \wedge \text{'each } \delta_i(\bar{z}, \bar{w}_i) \text{ is } K\text{-mutually algebraic'} \right)$$

As  $M \models \sigma$ , so does  $N$ . Choose  $\bar{c}^*$  from  $N$  so that  $\varphi(\bar{z})$  is  $Th(N_N)$ -equivalent to  $\theta(\bar{z}, \bar{c}^*)$  and  $\theta(\bar{z}, \bar{c}^*)$  is equivalent to a boolean combination of  $\delta_i(\bar{z}, \bar{c}_i) \in \mathcal{MA}(N)$ , where each  $\bar{c}_i$  is the corresponding subsequence of  $\bar{c}^*$ . Finally, rewrite  $\bar{z}$  as  $(\bar{x}, \bar{y})$  and substitute  $\bar{h}$  for  $\bar{y}$ . By Lemma 2.4(2), each of the formulas  $\delta_i(\bar{x}, \bar{h}, \bar{c}_i) \in \mathcal{MA}(N)$  and  $\varphi(\bar{x}, \bar{h})$  is  $Th(N_N)$ -equivalent to the boolean combination  $\theta(\bar{x}, \bar{h}, \bar{c}^*)$ . Thus,  $\varphi(\bar{x}, \bar{h}) \in \mathcal{MA}^*(N)$ , as required.

The following Lemma is folklore, but a proof is included for the convenience of the reader. Recall that a partitioned formula  $\varphi(\bar{x}, \bar{y})$  does not have the finite cover property (i.e., has nfcp) with respect to a theory  $T$  if there is a number  $k$  so that for all sets  $\{\bar{c}_i : i \in I\}$ , the type  $\Gamma := \{\varphi(\bar{x}, \bar{c}_i) : i \in I\}$  is consistent with  $T$  whenever every  $k$ -element subset of  $\Gamma$  is consistent with  $T$ .

**Lemma 2.11** *Let  $M$  be any structure, and let  $\varphi(\bar{x}, \bar{y})$  be any partitioned  $L(M)$ -formula. If, for some integer  $K$ , either  $M \models \forall \bar{y} \exists^{<K} \bar{x} \varphi(\bar{x}, \bar{y})$ , or  $M \models \forall \bar{y} \exists^{<K} \bar{x} \neg \varphi(\bar{x}, \bar{y})$ , then  $\varphi(\bar{x}, \bar{y})$  does not have the finite cover property with respect to  $Th(M_M)$ .*

**Proof.** If  $M$  is finite, then every partitioned formula  $\varphi(\bar{x}, \bar{y})$  has nfcp for trivial reasons, so assume that  $M$  is infinite. First, assume that  $M \models \forall \bar{y} \exists^{<K} \bar{x} \varphi(\bar{x}, \bar{y})$ . Choose tuples  $\{\bar{c}_i : i \in I\}$  from some elementary extension of  $M$  and assume that the type  $\Gamma := \{\varphi(\bar{x}, \bar{c}_i) : i \in I\}$  is inconsistent. It suffices to find a subtype of at most  $K$  elements that is inconsistent as well. Choose a maximal sequence  $\langle i_j : j \leq n \rangle$  from  $I$  such that  $i_0 \in I$  is arbitrary



and for each  $1 \leq m \leq n$ ,

$$\models \exists \bar{x} \left( \bigwedge_{j < m} \varphi(\bar{x}, \bar{c}_{i_j}) \wedge \neg \varphi(\bar{x}, \bar{c}_{i_m}) \right)$$

By our hypotheses on  $\varphi(\bar{x}, \bar{c}_{i_0})$ ,  $n \leq K$ . But now, if  $\bigwedge_{j \leq n} \varphi(\bar{x}, \bar{c}_{i_j})$  were consistent but  $\Gamma$  were not, we would contradict the maximality of the sequence.

In the other case, as  $M$  is infinite, every partial type of the form  $\{\varphi(\bar{x}, \bar{c}_i) : i \in I\}$  is consistent, so the nfcp of  $\varphi(\bar{x}, \bar{y})$  is vacuously true.

**Proposition 2.12** *For any structure  $M$ , the theory of the mutually algebraic reduct of  $M$  has nfcp.*

**Proof.** By the equivalence of (1) and  $\forall m(2)_m$  in Theorem II 4.4 of [6] (whose proof does not use stability) it suffices to show that no partitioned formula of the form  $\varphi(x, \bar{y}) \in \mathcal{MA}^*$  with  $\text{lg}(x) = 1$  has the finite cover property.

Consider any formula  $\theta(x, \bar{y})$  of the form

$$\bigwedge_{i < k} \beta_i(\bar{z}_i) \wedge \bigwedge_{j < m} \neg \gamma_j(\bar{z}_j)$$

with each  $\beta_i$  and  $\gamma_j$  from  $\mathcal{MA}$ . First, if the variable  $x$  occurs in any  $\beta_i$ , then it follows that there is a number  $K$  so that  $M \models \forall \bar{y} \exists^{<K} x \theta(x, \bar{y})$ . Second, if  $x$  does not occur in any  $\beta_i$ , then there is a number  $K$  so that there is a number  $K$  so that  $M \models \forall \bar{y} \exists^{<K} x \neg \theta(x, \bar{y})$ . But, any formula  $\varphi(x, \bar{y}) \in \mathcal{MA}^*$  is a finite disjunction of formulas  $\theta(x, \bar{y})$  described above. It follows that for some  $K$ , either  $M \models \forall \bar{y} \exists^{<K} x \varphi(x, \bar{y})$  or there is a number  $K$  so that  $M \models \forall \bar{y} \exists^{<K} x \neg \varphi(x, \bar{y})$  holds. Thus,  $\varphi(x, \bar{y})$  has the nfcp by Lemma 2.11.

### 3 Characterizing theories of mutually algebraic structures

We begin with two definitions indicating that the forking behavior of 1-types (types with a single free variable) is particularly simple.

**Definition 3.1** A complete, stable theory with an infinite model is *weakly minimal* if every forking extension of a 1-type is algebraic (equivalently if  $R^\infty(x = x) = 1$ ) and is *trivial* if there do not exist a set  $D$  and three elements  $\{a, b, c\}$  that are dependent, but pairwise independent over  $D$ . A type  $p \in S(D)$  is trivial if there do not exist a set  $\{a, b, c\}$  of realizations of  $p$  that are dependent, but pairwise independent over  $D$ .

It is well known that a weakly minimal theory is trivial if and only if every minimal type is trivial. The following Lemma generalizes the analogous result for non-trivial, strongly minimal theories that was proved by Baldwin and Baizhanov in [1].

**Lemma 3.2** *If  $T$  is weakly minimal and non-trivial, then there is a model  $M$  of  $T$  and a subset  $A \subseteq M$  such that  $(M, A)$  is unstable.*

**Proof.** Among all minimal types  $p \in S(D)$  and formulas  $\varphi(z, xy)$  over  $D$  that contain a dependent, but pairwise independent triple  $\{a, b, c\}$  of realizations of  $p$ , with the dependency witnessed by the algebraic formula  $\varphi(z, ab) \in \text{tp}(c/Dab)$ , choose one with the multiplicity of  $\varphi(z, ab)$  as small as possible. It follows from this multiplicity condition that  $\text{acl}(D \cup \{a\}) \cup \text{acl}(D \cup \{b\})$  does not contain any realizations of  $\varphi(z, ab)$ .

Fix  $p \in S(D)$  and  $\varphi(z, xy)$  as above, and let  $M$  be a sufficiently saturated model containing  $D$ . To ease notation, we may assume  $D = \emptyset$ . Let  $\langle (a_i, b_i) : i \in \omega \rangle$  be a Morley sequence in  $p^{(2)}$ . That is,  $\{a_i : i \in \omega\} \cup \{b_j : j \in \omega\}$  is an independent set of realizations of  $p$ . For each pair  $(i, j) \in \omega^2$ , choose  $c_{i,j} \in p(M)$  realizing  $\varphi(z, a_i b_j)$ . Let  $A = \{c_{i,j} : i \leq j < \omega\}$ . We argue that the  $L_P$ -formula  $\Phi(x, y) := \exists z(P(z) \wedge \varphi(z, xy))$  has the order property in  $(M, A)$ .

To see this, it is clear that the element  $c_{i,j}$  witnesses  $\Phi(a_i, b_j)$  whenever  $i \leq j$ . On the other hand, suppose some  $c_{k,\ell}$  witnessed  $\varphi(x, a_i, b_j)$ . We argue that we must have  $k = i$  and  $\ell = j$ : If neither equality held, then we would have  $c_{k,\ell}$  forking with both sets  $\{a_i, b_j\}$  and  $\{a_k, b_\ell\}$ . This is impossible, as the doubletons are independent from each other and the type  $p$  is minimal, hence regular, hence of weight one. Similarly, suppose that  $k = i$  but  $\ell \neq j$ . Then, working over  $a_i$ ,  $c_{i,\ell}$  is not algebraic over  $a_i$ , so  $\text{tp}(c_{i,\ell}/a_i)$  is parallel to  $p$ , hence is also regular, so of weight one. But, working over  $a_i$ ,  $c_{i,\ell}$  forks with each of  $b_j$  and  $b_\ell$ , which are independent over  $a_i$ . The case where  $j = \ell$  is symmetric, completing the proof.

In what follows, a *mutually algebraic expansion* of a structure  $M$  is an expansion formed by adding arbitrarily many new relation symbols  $R_i$ , whose interpretation is a mutually algebraic subset of  $M^{\text{arity}(R_i)}$  (see Definition 2.1). In the Theorem that follows, we do not require that the theory  $T$  be complete.

**Theorem 3.3** *The following are equivalent for any theory  $T$ :*

1. *Every model of  $T$  is a mutually algebraic structure;*
2. *Every mutually algebraic expansion of every model of  $T$  is a mutually algebraic structure;*
3.  *$\text{Th}((M, A))$  has the nfcp for every  $M \models T$  and every expansion  $(M, A)$  by a unary predicate;*
4. *Every complete extension of  $T$  having an infinite model is weakly minimal and trivial.*

**Proof.** (1)  $\Rightarrow$  (2) Fix  $M \models T$  and let  $\overline{M} = (M, R_i)_{i \in I}$  be any expansion of  $M$ , where each  $R_i$  is a  $k(i)$ -ary relation symbol whose interpretation in  $\overline{M}$  is a mutually algebraic subset  $B_i \subseteq M^{k(i)}$ . By definition, the  $\overline{M}$ -definable subsets are the smallest class of subsets of  $M^\ell$  for various  $\ell$  that contain every  $M$ -definable set and every  $B_i$  and are closed under boolean combinations and projections. As  $M$  is mutually algebraic, every  $M$ -definable set is a boolean combination of mutually algebraic sets. So  $\mathcal{MA}^*(\overline{M})$  contains every  $M$ -definable set and each of the sets  $B_i$ . Additionally,  $\mathcal{MA}^*(\overline{M})$  is closed under boolean combinations and projections. Thus, every  $\overline{M}$ -definable set is in  $\mathcal{MA}^*(\overline{M})$ , so  $\overline{M}$  is a mutually algebraic structure.

(2)  $\Rightarrow$  (3) Fix any  $M \models T$  and any expansion  $\overline{M} = (M, A)$  by a unary predicate. As every subset of  $M^1$  is mutually algebraic, it follows from (2) that  $\overline{M}$  is a mutually algebraic structure, i.e., every  $\overline{M}$ -definable set is in  $\mathcal{MA}^*(\overline{M})$ . Thus, every partitioned  $\overline{M}$ -definable formula  $\varphi(\bar{x}, \bar{y})$  has nfcp by Proposition 2.12. That is, the elementary diagram of  $\overline{M}$  and hence the theory of  $\overline{M}$  has nfcp.

(3)  $\Rightarrow$  (4) Suppose  $T$  satisfies (3). Fix any complete extension  $T'$  of  $T$  with an infinite model. As the nfcp implies stability,  $T'$  must be stable. Fix a sufficiently saturated model  $M$  of  $T'$ . As  $T'$  is stable, if it were not weakly minimal then we could choose an element  $a$  and a tuple  $\bar{b}$  from  $M$  such that

$\text{tp}(a/\bar{b})$  forks over the empty set, but  $a$  is not algebraic over  $\bar{b}$ . Let  $\varphi(x, \bar{y})$  be chosen so that  $\varphi(x, \bar{b}) \in \text{tp}(a/\bar{b})$  witnesses the forking. As  $M$  is sufficiently saturated, we can find a Morley sequence  $\langle \bar{b}_i : i \in \omega \rangle$  in  $\text{stp}(\bar{b})$  inside  $M$ . As  $T'$  is stable,  $\{\bar{b}_i : i \in \omega\}$  is an indiscernible set and there is a number  $k$  so that every element  $a^* \in M$  is contained in at most  $k$  of the sets  $D_i := \varphi(M, \bar{b}_i)$ . As each  $D_i$  is infinite, we can construct a subset  $A$  of  $M$  such that each  $c \in A$  is contained in exactly one of the sets  $D_i$ , and for each  $i$ ,  $|A \cap D_i| = i$ . Then the theory of the expansion  $(M, A)$ , where the new unary predicate symbol  $P$  is interpreted as  $A$ , has the finite cover property as witnessed by the  $L_P$ -formula  $\Psi(x, \bar{y}z) := P(x) \wedge \varphi(x, \bar{y}) \wedge x \neq z$ . Thus,  $T$  must be weakly minimal. That  $T'$  must be trivial as well follows from Lemma 3.2 and the fact that instability implies an instance of the finite cover property.

(4)  $\Rightarrow$  (1) This is the content of Theorem 4.2 of [5]. In fact, there it is shown that every  $M$ -definable formula is a boolean combination of mutually algebraic formulas of a very special form.

**Corollary 3.4** *Let  $M$  be any infinite structure. The mutually algebraic reduct of  $M$  described in Definition 2.9 is the maximal weakly minimal, trivial reduct of  $M$ .*

**Proof.** The mutually algebraic reduct of  $M$  is a mutually algebraic structure, so it has a weakly minimal, trivial theory. Conversely, if any reduct of  $M$  has a weakly minimal, trivial theory, then it is a mutually algebraic structure, hence all of its definable sets are contained in  $\mathcal{MA}^*(M)$ .

## 4 Mutually algebraic structures

Suppose that  $M$  is a mutually algebraic structure in a language  $L$ . We study models of the elementary diagram of  $M$ , or equivalently the class of elementary extensions of  $M$ . Note that if  $M$  is finite, then there are no proper elementary extensions of  $M$ , which will render all of the results that follow vacuous. Because of this, **throughout this section we additionally assume that  $M$  is infinite**. Thus, we may assume that  $M$  is elementarily embedded in a much larger, saturated ‘monster model’  $\mathfrak{C}$ .

By Theorem 3.3,  $Th(M)$  is weakly minimal and trivial, so the quantifier elimination offered in [5] applies. Specifically, let

$\mathcal{A}(M) := \{\text{all quantifier-free mutually algebraic } L(M)\text{-formulas } \alpha(\bar{z})\}$  and

$\mathcal{E}(M) = \{\text{all } L(M)\text{-formulas of the form } \exists \bar{x} \alpha(\bar{x}, \bar{y}), \text{ where } \alpha(\bar{x}, \bar{y}) \in \mathcal{A}(M)\}$

and let  $\mathcal{A}^*(M)$  (respectively  $\mathcal{E}^*(M)$ ) denote the closure of  $\mathcal{A}(M)$  (respectively  $\mathcal{E}(M)$ ) under boolean combinations. Proposition 4.1 of [5] states that every quantifier-free  $L(M)$ -formula is equivalent to a formula in  $\mathcal{A}^*(M)$ , while Theorem 4.2 states that every  $L(M)$ -formula is equivalent to a formula in  $\mathcal{E}^*(M)$ .

As  $Th(M)$  is weakly minimal, the relation ‘ $a \in \text{acl}_M(B)$ ’ satisfies the axioms of a pre-geometry, where  $\text{acl}_M(B)$  abbreviates  $\text{acl}(M \cup B)$ . (Algebraic closures are always computed with respect to satisfaction in  $\mathfrak{C}$ .) Thus, the binary relation  $a \approx b$  on  $\mathfrak{C} \setminus M$  defined by  $a \in \text{acl}_M(\{b\})$  is an equivalence relation. The following easy Lemma is folklore.

**Lemma 4.1** *Suppose  $Th(M)$  is weakly minimal,  $M \preceq \mathfrak{C}$ , and  $A$  is any algebraically closed set satisfying  $M \subseteq A \subseteq \mathfrak{C}$ . Then  $A$  is the universe of an elementary submodel of  $\mathfrak{C}$ .*

**Proof.** The interpretation of any constant symbol is contained in  $M$ , and the fact that  $A$  is algebraically closed implies that it is closed under every function symbol in the language. Thus,  $A$  is the universe of a substructure of  $\mathfrak{C}$ . To see that this substructure is elementary, by the Tarski-Vaught criterion it suffices to show that for any  $L$ -formula  $\varphi(x, \bar{y})$  and for any  $\bar{a}$  from  $A$ , if  $\mathfrak{C} \models \exists x \varphi(x, \bar{a})$ , then there is  $b \in A$  such that  $\mathfrak{C} \models \varphi(b, \bar{a})$ . So fix any  $\varphi(x, \bar{a})$  and  $b \in \mathfrak{C}$  such that  $\mathfrak{C} \models \varphi(b, \bar{a})$ . If  $b \in A$ , then we are done, so assume  $b \notin A$ . As  $A$  is algebraically closed, this means that  $\text{tp}(b/M\bar{a})$  is not algebraic. As  $Th(M)$  is weakly minimal and  $b$  is a singleton, this implies that  $\text{tp}(b/M\bar{a})$  does not fork over  $M$ . But then, by symmetry and finite satisfiability of non-forking over models, there is  $b^* \in M \subseteq A$  such that  $\mathfrak{C} \models \varphi(b^*, \bar{a})$ .

Recall that when combined with weak minimality, triviality implies that for any set  $B \subseteq \mathfrak{C}$ ,  $\text{acl}_M(B) = \bigcup_{b \in B} \text{acl}_M(\{b\})$ .

**Proposition 4.2** *Let  $M$  be any mutually algebraic  $L$ -structure.*

1. If  $M \subseteq A \subseteq \mathfrak{C}$  and  $A$  is an arbitrary union of  $\approx$ -classes, then  $A$  is an  $L$ -structure and  $M \preceq A \preceq \mathfrak{C}$ ; and
2. Conversely, if  $M \preceq N \preceq \mathfrak{C}$  and  $B \subseteq N \setminus M$  is a set of  $\approx$ -representatives, then  $N$  is the disjoint union of the sets  $M$  and  $\{\text{acl}_M(\{b\}) \setminus M : b \in B\}$ .

**Proof.** (1)  $Th(M)$  is weakly minimal and trivial by Theorem 3.3. By triviality,  $A$  must be algebraically closed, so  $A \preceq \mathfrak{C}$  by Lemma 4.1. That  $M \preceq A$  follows immediately from this.

(2) That the sets are disjoint follows by triviality. If there were an element  $d \in N$  that was not in any of these sets, then  $d$  would be  $\approx$ -inequivalent to every element of  $B$ , contradicting the maximality of  $B$ .

In light of the previous Proposition, it is natural to refer to the sets  $\text{acl}_M(\{b\}) \setminus M$  as the *components* of a given  $N \succeq M$ . Each component has size bounded by the number of  $L(M)$ -formulas, and one can speak of the type of a fixed enumeration of a component over  $M$ . The notion of a component map records this amount of data.

**Definition 4.3** Suppose that  $M$  is a mutually algebraic structure and  $N_1, N_2$  are both elementary extensions of  $M$ . A *component map*  $f : N_1 \rightarrow N_2$  is a bijection such that  $f|_M = id$  and for each  $b \in N_1 \setminus M$ ,

- $f$  restricted to  $\text{acl}_M(\{b\})$  is elementary and
- $f(\text{acl}_M(\{b\})) = \text{acl}_M(\{f(b)\})$  setwise.

**Proposition 4.4** Suppose that  $M$  is mutually algebraic and  $N_1, N_2 \succeq M$ . Then every component map  $f : N_1 \rightarrow N_2$  is an isomorphism. Conversely, every isomorphism  $f : N_1 \rightarrow N_2$  that is the identity on  $M$  is a component map.

**Proof.** As every quantifier-free  $L(M)$ -formula is equivalent to a formula in  $\mathcal{A}^*(M)$ , it suffices to show that  $f$  preserves every formula  $\alpha(\bar{z}) \in \mathcal{A}(M)$ . Choose any  $\bar{a}$  from  $N_1$ . Without loss, by Lemma 2.4(2) and the fact that  $f$  fixes  $M$  pointwise, we may assume  $\bar{a}$  is disjoint from  $M$ . There are now two cases. First, if  $\bar{a} \subseteq \text{acl}_M(\{b\})$  for some element  $b$ , then  $f(\bar{a}) \subseteq \text{acl}_M(\{b\})$  and  $N_1 \models \alpha(\bar{a})$  if and only if  $N_2 \models \alpha(f(\bar{a}))$  by the elementarity of  $f$  restricted to  $\text{acl}_M(\{b\})$ . Second, if  $\bar{a}$  intersects at least two components, then  $N_1 \models \neg\alpha(\bar{a})$

automatically. Furthermore, since  $f$  maps components onto components,  $f(\bar{a})$  would intersect at least two components of  $N_2$ , so  $N_2 \models \neg\alpha(\bar{a})$ . Thus,  $\alpha$  is preserved in both cases, so  $f$  is an isomorphism. The converse is clear since elementary maps preserve algebraic closure.

We close with two examples of how the analog of Proposition 4.4 can fail if we work over  $\text{acl}(\emptyset)$  instead of a model.

**Example 4.5** Let  $L = \{R, S, E\}$ , and let  $T$  be the theory asserting that  $E$  is an equivalence relation with exactly two classes, both infinite, and  $R$  is a binary ‘mating relation’ i.e.,  $R$  is symmetric, irreflexive, and  $\forall x \exists^=1 y R(x, y)$ . We further require that  $R(x, y) \rightarrow \neg E(x, y)$ . Take  $S$  to be a 4-ary relation such that  $S(x, y, z, w)$  holds if and only if the four elements are distinct, and each of the relations  $R(x, y)$ ,  $R(z, w)$ , and  $E(x, z)$  hold. Then  $T$  is complete, mutually algebraic, and  $\text{acl}(\emptyset) = \emptyset$ . For any model  $N$  of  $T$ , the decomposition of  $N$  into  $R$ -mated pairs is a decomposition of  $N$  into two-element ‘ $\emptyset$ -components’ i.e., sets  $A$  satisfying  $\text{acl}_\emptyset(A) = A$ . However, in contrast to Proposition 4.4, there are ‘ $\emptyset$ -component maps’  $f : N \rightarrow N$ , i.e., bijections  $f : N \rightarrow N$  whose restriction to each two-element  $\emptyset$ -component is elementary, that are not automorphisms.

The second example is from [3]. There, Baldwin, Shelah, and the author exhibit two models  $M, N$  of the theory of infinitely many, binary splitting equivalence relations that are not isomorphic in the set-theoretic universe  $V$ , but there is a c.c.c. extension  $V[G]$  of  $V$  and  $M \cong N$  in  $V[G]$ . This theory is also weakly minimal and trivial with  $\text{acl}(\emptyset) = \emptyset$ . In fact, this theory has a prime model and every ‘component’ is a singleton. The complexity exploited by this example involves which strong types over the empty set are realized in the models  $M$  and  $N$ .

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